Chy 475—Spring 2009 **Angular momentum in quantum mechanics**
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Introduction to angular momentum

Classically, angular momentum is a vector $\vec{L} = \vec{r} \times \vec{p}$ where the the vector $\vec{L}$ is perpendicular to the plane containing r and p. Use the right-hand-rule to determine the direction of $\vec{L}$. The magnitude of $\vec{L}$ is $|L| = I\omega$ where I is the moment of inertia of the particle about the chosen axis and ω is the angular velocity in radians/second. To get the components of $\vec{L}$ one can use the following relation using the properties of the determinant:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = i(yp_z - zp_y) + j(zp_x - xp_z) + k(xp_y - yp_x)$$

So the components of angular momentum are, in terms of the components of $\vec{r}$ and $\vec{p}$:

$$L_x = (yp_z - zp_y) \quad \rightarrow \quad \hat{L}_x = (y\hat{p}_z - z\hat{p}_y) = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$$

$$L_y = (zp_x - xp_z) \quad \rightarrow \quad \hat{L}_y = (z\hat{p}_x - x\hat{p}_z) = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$$

$$L_z = (xp_y - yp_x) \quad \rightarrow \quad \hat{L}_z = (x\hat{p}_y - y\hat{p}_x) = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

Here the quantum mechanical operators for angular momentum are given by associating substituting the corresponding momentum operators.

We've seen that in spherical polar coordinates, the square of the angular momentum is intimately related to the rigid-rotor kinetic energy, $E_{rot} = I\omega^2/2 = L^2/2I$, and to the Hamiltonian for the hydrogen atom:

$$\hat{H}^{el} = -\frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\hat{L}^2\right] - \frac{e^2}{4\pi\varepsilon_0 r}$$

We've also seen that the hydrogen atom eigenstates are labeled by three quantum numbers, *n*, *l*, and *m* of which only *n*, the principal quantum number affects the energy of the state, giving n^2 degenerate orbitals for each level.

Angular momentum eigenstates are the spherical harmonics

We can use the properties of the dot product to show that

$$\hat{L}^2 = \hat{L} \cdot \hat{L} = -\hbar^2\left(\hat{L}_x + \hat{L}_y + \hat{L}_z\right) \cdot \left(\hat{L}_x + \hat{L}_y + \hat{L}_z\right) = -\hbar^2\left(\hat{L}_x^{\ 2} + \hat{L}_y^{\ 2} + \hat{L}_z^{\ 2}\right)$$

It is often more useful to have the relationship between the components of the angular momentum and the derivatives with respect to θ and ϕ:

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \quad ; \hat{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}) \ ; \hat{L}_y = i\hbar(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi})$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

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A. Verify that Y_{10}, Y_{11} are eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$ respectively.

B. Write out the equations for the expectation values of $\hat{L}^2$ and $\hat{L}_z$ in the states Y_{10}, Y_{11}, and Y_{32}? Give the values.

C. Give the magnitude of $\langle \hat{L}^2 \rangle$, $\langle \hat{L}_z \rangle$, and $|L| = \sqrt{\langle \hat{L}^2 \rangle}$ for the states of part B.