 Commuting operators & simultaneous eigenfunctions
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Some operators don't commute

For real numbers, the property of multiplication is commutative since aXb = bXa or aXb - bXa =0. We define the commutator of two quantum mechanical operators as follows:

$$\left[\hat{A},\hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$

These operators are understood to be operating on a function so we could write,

$$\left[\hat{A},\hat{B}\right]f = \hat{A}\hat{B}f - \hat{B}\hat{A}f = \hat{A}\left(\hat{B}f\right) - \hat{B}\left(\hat{A}f\right)$$

There are two possibilities:

1) the commutator is zero; in this case the operators $\hat{A}$ and $\hat{B}$ are said to commute
2) the commutator is non-zero; in this case the operators $\hat{A}$ and $\hat{B}$ don't commute.

A. Show that the two operators $\hat{x}$ and $\hat{p}_z = -i\hbar\frac{\partial}{\partial z}$ commute by forming the commutator and allowing it to act on an arbitrary function *f(x)*. What if the function is *f(z)?*

B. Show that the two operators $\hat{x}$ and $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$ do not commute by forming the commutator and allowing it to act on an arbitrary function *f(x)*.

Theorem 1*: If two quantum mechanical operators commute, they share the same set of eigenfunctions. If two operators share the same set of eigenfunctions, they must commute.*

A consequence of this theorem is that if the operators have simultaneous eigenfunctions, we can assign definite eigenvalues to both operators and thus can measure the value of the corresponding property at the same time.

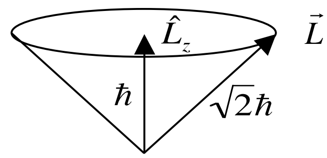
On the other hand, if the two operators don't commute, we can't assign definite values to these observables at the same time. In the case of $\hat{x}$ and $\hat{p}_x$ for example this means that we can know the position or the momentum exactly but not both at the same time. This is another way of stating the Heisenberg uncertainty principle.

Commutators for angular momentum

$$\left[\hat{L}^2,\hat{L}_x\right]=0 \qquad \left[\hat{L}^2,\hat{L}_y\right]=0 \qquad \left[\hat{L}^2,\hat{L}_z\right]=0$$
$$\left[\hat{L}_x,\hat{L}_y\right]=i\hbar L_z \qquad \left[\hat{L}_y,\hat{L}_z\right]=i\hbar L_x \qquad \left[\hat{L}_z,\hat{L}_x\right]=i\hbar L_y$$

These results imply that we can know the value of $\hat{L}^2$ and any one of its components simultaneously but not all three. We conventionally choose to know $\hat{L}_z$.

Consider a hydrogen atom in the state, ψ_{211}. For this state l=1 and m=+1. We can draw the vectors that represent the angular momentum and its z-component as follows



Note that the magnitude of $\hat{L}^2$ (a scalar) is known ($\left\langle \hat{L}^2 \right\rangle = 2\hbar^2$) as is the magnitude of the angular momentum vector ($|L| = \sqrt{2}\hbar$) and the value of the z-component ($\left\langle \hat{L}_z \right\rangle = \hbar$). However, the x- and y-components of the angular momentum cannot be determined at the same time as $\left\langle \hat{L}_z \right\rangle$ so our diagram shows a circle describing the possible positions of the vector $\vec{L}$ which must lie on the cone indicated.

C. Show that the angle between the vector $\vec{L}$ and the z-axis is π/4 radians or 45°.

We can think of sets of commuting operators as providing labels for states in quantum mechanics. Thus, for the hydrogen atom, $\hat{H}^{el}$, $\hat{L}^2$, and $\hat{L}_z$, all commute with each other and provide the simultaneous expectation values of energy, angular momentum squared and the z-axis projection of angular momentum, labeled by the quantum numbers, n, l, and m (sometimes called m_l).